ON SERIES IN LINEAR TOPOLOGICAL SPACES*

BY

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ABSTRACT

The main result is that in every complete locally-bounded linear topological space there exist series which are unconditionally yet not absolutely convergent. Relations between absolute, unconditional and metric convergence of series are studied.

1. **Introduction.** The primary purpose of this paper is to study the connection between absolute and unconditional convergence of series in general linear topological spaces (over the real or complex field). Since our auxiliary considerations involve metric spaces we study also the connections with metric convergence. There being no universal agreement about the terminology we explain the above terms at the outset.

A series

$$\sum_{n=1}^{\infty} x_n \tag{1.1}$$

in a linear topological space is said to be *absolutely convergent* (A) if

$$\sum_{n=1}^{\infty} M_V(x_n) < \infty \tag{1.2}$$

for every neighborhood V of the origin, where $M_V(x)$ is the Minkowski functional of x relative to V, i.e.

$$M_V(x) = \inf\{\lambda : \lambda > 0, \quad x \in \lambda V\}. \tag{1.3}$$

A series (1.1) in a linear topological space is said to be *unconditionallly*, or commutatively, *convergent* (U) if every series obtained from (1.1) by rearrangement is convergent.

In complete spaces this is equivalent to the requirement that (1.1) is sub-series convergent, i.e. that $\sum_{\nu=1}^{\infty} x_{n_\nu}$ be convergent for every strictly increasing sequence of positive integers n_ν.

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In complete locally convex spaces $A \Rightarrow U$. Both these notions are topological and the properties of absolute and unconditional convergence are preserved under isomorphisms.

The third kind of convergence we consider is of a different nature.

A series (1.1) in a linear metric space is said to be *metrically convergent* (M) if

$$\sum_{n=1}^{\infty} d(x_n, 0) < \infty, \tag{1.4}$$

where $d(x, y)$ denotes the distance between x and y.

This is very far from being a topological notion. Even one dimensional space can be metrized in such a way that M does not imply even ordinary convergence; or also so that A does not imply M. Nevertheless it will prove very useful for studying the relations between the topological notions A and U.

If the metric in the space is *translation invariant* we write $\| x \| = d(x, 0)$ and (1.4) becomes

$$\sum_{n=1}^{\infty} \| x_n \| < \infty. \tag{1.5}$$

(In general this 'norm' is not linear homogeneous. $\| \lambda x \|$ is continuous in both variables but need not even be monotone in λ for $\lambda > 0$.)

In Banach spaces (1.5) is equivalent to the absolute convergence of (1.1). In arbitrary complete metric spaces with a translation invariant metric (F^* spaces) we still have $M \Rightarrow A$, but the converse need not be true.

It is well known that $A \Leftrightarrow U$ in finite dimensional spaces. C.A. Rogers and the author proved [4] that this is not the case in any infinite dimensional Banach space, i.e. in every such space there exist series which are U but not A. This is equivalent to saying that in every infinite dimensional Banach space there exist series which are U but not M. This last statement was generalized from Banach spaces to arbitrary F^* spaces by S. Rolewicz(1) [8]. Since A need not imply M in these spaces it does not entail the existence of series which are U but not A, and indeed in the space of all sequences $x = (\xi_1, \ldots, \xi_n, \ldots)$ with the topology of coordinate convergence(2) $A \Leftrightarrow U$ (see e.g. [1] p. 63).

Our main result is that in *every* infinite-dimensional locally-bounded(3) complete

(1) Employing an older terminology S. Rolewicz calls M absolute convergence. The author in [3] also uses the older terminology. The present paper was announced in [3] as forthcoming under a different title (On series in Fréchet spaces.)

(2) This space can be metrized by

$$\| x \| = \sum_{n=1}^{\infty} 2^{-n} |\xi_n| (1 + |\xi_n|)^{-1}.$$

(3) This means that there exists a neighborhood of the origin which is contained in a sufficiently large homothetic image of any other given neighborhood of the origin.

linear topological space U *does not imply* A. This seems a rather surprising generalization of the result about Banach spaces. In the opposite direction we show that every linear metrizable topological space containing arbitrarily short rays([4]) has an infinite dimensional subspace in which U and A are equivalent (we construct a subspace isomorphic to the one given in footnote([2])). It is remarkable that local convexity does not enter into these statements.

2. **Statement of results.** We begin with results about metric convergence. For a linear metric space we put

$$\Delta(X) = \sup_{x \in X} d(x,0). \tag{2.1}$$

THEOREM 1. *If X is a complete linear metric space containing arbitrarily short rays and (γ_n) is a sequence of positive numbers satisfying*

$$0 < \gamma_n < \Delta(X) \qquad (n = 1,2,\ldots) \tag{2.2}$$

and

$$\lim_{n=\infty} \gamma_n = 0 \tag{2.3}$$

then there exists in X an unconditionally convergent series (1.1) *with*

$$d_n(x,0) = \gamma_n \qquad (n = 1,2,\ldots). \tag{2.4}$$

In order to state concisely the next results we introduce the following definitions.

Let X be a linear metric space and $\rho > 0$. We put

$$B^*(\rho) = \{x : d(\mu x, 0) \leqq \rho \qquad \text{for} \quad 0 \leqq \mu \leqq 1\}. \tag{2.5}$$

$B^*(\rho)$ is a closed star-shaped neighborhood of the origin. The sets $B^*(\rho)$ obviously constitute a fundamental system of neighborhoods of the origin.

Let $\phi(\lambda)$ be any positive function defined for $0 < \lambda < 1$. We say that X *has property* $C^*(\phi)$ if

$$B^*(\lambda\rho) \subset \phi(\lambda) B^*(\rho) \qquad \text{for all } \rho > 0 \text{ and } 0 < \lambda < 1. \tag{2.6}$$

The following may be considered the key result of the paper.

THEOREM 2. *Let X be an infinite-dimensional complete linear metric space having the property $C^*(\phi)$ and let $\gamma_n (n = 1,2,\ldots)$ be a sequence satisfying* (2.2) *and*

$$\sum_{n=1}^{\infty} \phi^2(\sigma\gamma_n) < \infty \qquad \text{for every } 0 < \sigma < \infty. \tag{2.7}$$

([4]) This means that to every neighborhood of the origin there corresponds some $x \neq 0$ for which the whole ray λx, ($\lambda \geqq 0$), is contained in it.

Then there exists in X an unconditionally convergent series (1.1) *for which* (2.4) *holds.*

An immediate consequence is the following

COROLLARY 1. *Let X be an infinite dimensional complete linear metric space with a translation invariant metric and let $\gamma_n(n = 1,2,\ldots)$ be a sequence satisfying* (2.2) *and*

$$\sum_{n=1}^{\infty} \gamma_n^2 < \infty. \tag{2.8}$$

Then there exists in X an unconditionally convergent series (1.1) with

$$\| x_n \| = \gamma_n, \qquad (n = 1,2,\ldots). \tag{2.9}$$

Indeed, this result follows at once from the observation that if the metric is translation invariant then X has the property $C^*(\phi)$ with $\phi(\lambda) = 2\lambda$. This is seen as follows: If $x \in B^*(\lambda\rho)$ with $0 < \lambda < 1$, then for every $0 \leqq \mu \leqq 1$ we have[5]

$$\left\| \mu\frac{x}{2\lambda} \right\| = \left\| \frac{\mu}{2\lambda[1/\lambda]}\left[\frac{1}{\lambda}\right]x \right\| \leqq \sum_{i=1}^{[1/\lambda]} d\left(\frac{i\mu x}{2\lambda[1/\lambda]}, \frac{(i-1)\mu x}{2\lambda[1/\lambda]}\right)$$
$$= \left[\frac{1}{\lambda}\right] \left\| \frac{\mu}{2\lambda[1/\lambda]}x \right\| \leqq \left[\frac{1}{\lambda}\right] \lambda\rho \leqq \rho,$$

i.e. $(x/2\lambda) \in B^*(\rho)$ as asserted.

For Banach spaces this corollary is precisely the main result of [4]. It is again noteworthy that local convexity is not needed. Taking e.g. $\gamma_n = 1/n$ (for $n > 1/\Delta(X)$) we obtain the result of S. Rolewicz quoted in the introduction.

As another consequence of Theorem 2 we deduce our principal result of a topological nature.

THEOREM 3. *In every infinite-dimensional locally-bounded complete linear space there exist unconditionally convergent series which are not absolutely convergent.*

In the converse direction we prove

THEOREM 4. *Every metrizable complete linear space containing arbitrarily short rays has infinitely dimensional subspace in which every unconditionally convergent series is absolutely convergent.*

We conclude with a result showing how little connection there is between M and A even in F^* spaces.

THEOREM 5. *Let X be a metrizable linear topological space and $\gamma_n(n = 1,2,\ldots)$ any sequence of positive numbers satisfying* (2.3). *Then there exists a translation invariant metrization of X (reproducing, of course, the given topology) such that* (1.1) *is absolutely convergent whenever*

(5) Square brackets denote the integral part.

$$\|x_n\| \leqq \gamma_n \qquad (n = 1,2,\ldots). \tag{2.10}$$

3. **Proof of Theorem** 1. Given $\alpha > 0$ there exists $\beta > 0$ such that $d(x,0) < \beta$ and $d(y,0) < \beta$ together imply $d(x + y,0) < \alpha$. Thus starting with any $\alpha_0 > 0$ we can define consecutively a sequence of positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots$ satisfying

(3.1) $d(x + y,0) \leqq \alpha_{\nu-1}$ whenever $d(x,0) \leqq \alpha_\nu$ and $d(y,0) \leqq \alpha_\nu$ $(\nu = 1,2,\ldots)$.

We may moreover assume

$$\lim_{\nu=\infty} \gamma_\nu = 0. \tag{3.2}$$

(This is automatic if $\alpha_0 < \Delta(X)$, then $\alpha_\nu < 2^{-\nu}\alpha_0$).

Put

$$\Delta(x) = \sup_{\lambda>0} \quad d(\lambda x,0). \tag{3.3}$$

Since X contains arbitrarily short rays there exist non zero y_ν in X satisfying

$$\Delta(y_\nu) < \alpha_\nu \quad (\nu = 1,2,\ldots). \tag{3.4}$$

Finally we determine a strictly increasing sequence of positive integers k_ν such that

$$\gamma_n < \Delta(y_\nu) \quad \text{for} \quad n \geqq k_\nu \quad (\nu = 1,2,\ldots). \tag{3.5}$$

We now define x_n satisfying (2.4) as follows: Arbitrarily for $1 \leqq n < k_1$ and as a positive multiple of y_ν for $k_\nu \leqq n < k_{\nu+1}$ $(\nu = 1,2,\ldots)$. This can be done because $d(\lambda x,0)$ is continuous in λ.

We claim that the series thus obtained is unconditionally convergent. Indeed, denoting by a prime summation over any assigned subseries we have for $m \geqq k_i$

$$\sum_{n=k_i}^{m}{}' x_n = \sum_{\nu=i}^{j-1} \sum_{n=k_\nu}^{k_{\nu+1}-1}{}' x_n + \sum_{n=k_j}^{m}{}' x_n \tag{3.6}$$

where j is determined by $k_j \leqq m < k_{j+1}$. By our construction, (3.3) and (3.4), we have

$$d\left(\sum_{n=k_\nu}^{k_{\nu+1}-1}{}' x_n, 0\right) \leqq \Delta(y_\nu) < \alpha_\nu, \quad d\left(\sum_{n=k_j}^{m}{}' x_n, 0\right) \leqq \Delta(y_j) < \alpha_j. \tag{3.7}$$

Applying (3.1) from the last summand of (3.6) backwards we obtain

$$d\left(\sum_{n=k_i}^{m}{}' x_n, 0\right) \leqq \alpha_{i-1}.$$

As this holds for any $m \geqq k_i$ it follows, by (3.2), that $\Sigma' x_n$ is Cauchy convergent; hence, by completeness, convergent. This being true for any subseries, the theorem is established.

4. A geometrical lemma.

LEMMA 1. *Let B be a compact set in Euclidean m-space containing the origin as an interior point. Let $\| \ \|$ denote the Euclidean norm, relative to a given orthonormal set $e_1, \ldots, e_m$. There exists a centro-affine transformation T having the properties*

$$\{x : \| x \| \leqq 1\} \subset TB \tag{4.1}$$

and there are points $p_1, \ldots, p_m$ on the boundary of TB such that

$$p_j = \sum_{i=1}^{j} \pi_{j,i} \, e_i \qquad (j = 1, \ldots, m) \tag{4.2}$$

with

$$\sum_{i=1}^{j-1} \pi_{j,i}^2 = 1 - \pi_{j,j}^2 \leqq \frac{j-1}{m}. \tag{4.3}$$

Specialized to convex symmetrical B this is precisely the fundamental lemma of [4]. The proof, however, does not appeal at all to the symmetry or convexity of B and holds verbatim for any compact B containing the origin as an interior point (it is even possible to relax somewhat the requirement of compactness). Since the proof is reproduced in Day's book [1] (p. 61) we shall not repeat it here.

For any real $\lambda_1, \ldots, \lambda_k$ $(1 \leqq k < m)$ we have

$$\sum_{j=1}^{k} \lambda_j p_j = \sum_{j=1}^{k} \lambda_j \pi_{j,j} e_j + \sum_{j=1}^{k} \lambda_j (p_j - \pi_{j,j} e_j). \tag{4.4}$$

By (4.2) and (4.3)

$$\Big\| \sum_{j=1}^{k} \lambda_j \pi_{i,j} e_j \Big\| \leqq \Big(\sum_{j=1}^{k} \lambda_j^2 \Big)^{1/2} \tag{4.5}$$

and

$$\Big\| \sum_{j=1}^{k} \lambda_j (p_j - \pi_{j,j} e_j) \Big\| \leqq \sum_{j=1}^{k} |\lambda_j| \left(\frac{j-1}{m} \right)^{1/2} \leqq \Big(\sum_{j=1}^{k} \lambda_j^2 \Big)^{1/2} \left(\sum_{j=1}^{k} \frac{j-1}{m} \right)^{1/2}. \tag{4.6}$$

Combining (4.4), (4.5) and (4.6) we obtain[6]

$$\Big\| \sum_{j=1}^{k} \lambda_j p_j \Big\| \leqq \left\{ 1 + \left(\frac{k(k-1)}{2m} \right)^{1/2} \right\} \Big(\sum_{j=1}^{k} \lambda_j^2 \Big)^{1/2}. \tag{4.7}$$

Taking this into account we may reformulate the above lemma as follows:

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(6) A similar computation leading to a somewhat weaker estimate occurs in [4] (and [1]). The improvement is of no consequence for the present paper and is brought for future reference. An occasionally better estimate is given by A. Grothendieck [5] but there is a mistake in his derivation (since the exact nature of the estimate is of no importance in [5] none of its results are affected).

LEMMA 2. *Let B be a compact set containing the origin as an interior point in real m-dimensional space. Then a Euclidean norm $\| \quad \|$ may be introduced into the space so that*

$$\{x : \| x \| \leqq 1\} \subset B \tag{4.8}$$

and there exist points $p_1, \ldots, p_m$ of unit norm on the boundary of B for which (4.7) *holds for all real $\lambda_1, \ldots, \lambda_k$ $(1 \leqq k \leqq m)$.*

5. **Proof of Theorem** 2. In view of Theorem 1 we may assume that X does not contain arbitrarily short rays. Then, cf. (3.3),

$$\delta = \inf_{x \in X} \Delta(x) > 0 \tag{5.1}$$

and we may start with $x_0 < \delta$ and construct a sequence of positive numbers $\alpha_1, \alpha_2, \ldots, \alpha_\nu, \ldots$ satisfying (3.1).

Let $0 < \rho_1 < \alpha_1$ and let k_1 be such that

$$\sum_{n=k_1}^{\infty} \phi^2(\gamma_n/\rho_1) < \frac{1}{4}\,.$$

Having defined ρ_ν and k_ν $(\nu = 1, 2, \ldots)$ we choose $\rho_{\nu+1}$ and then $k_{\nu+1} > k_\nu$ so that

$$0 < \rho_{\nu+1} < \alpha_{\nu+1} \text{ and } \sum_{n=k_\nu+1}^{\infty} \phi^2\left(\frac{\gamma_n}{\rho_{\nu+1}}\right) < \frac{1}{4}\,. \tag{5.2}$$

The existence of such k_ν is assured by (2.7).

We now proceed to define the x_n in (1.1). For $1 \leqq n < k_1$ we choose them arbitrarily subject to (2.4).

For $k_\nu \leqq n < k_{\nu+1}$ we proceed as follows: Put $k = k_{\nu+1} - k_\nu$ and let Y be an $m = k^2$ dimensional subspace of X. $B = Y \cap B^*(\rho_\nu)$ is compact (since $\rho_\nu < \delta$) and contains the origin as an interior point. Let $\| \quad \|$ be the Euclidean norm in Y and p_j the points whose existence is asserted in Lemma 2. Then we have

$$\left\| \sum_{j=1}^{k} \lambda_j p_j \right\| \leqq \left\{ 1 + \left(\frac{k-1}{2k} \right)^{1/2} \right\} \left(\sum_{j=1}^{m} \lambda_j^2 \right)^{1/2} \tag{5.3}$$

for all real $\lambda_1, \ldots, \lambda_k$.

The points p_j may not quite do for our purpose. It may happen that though they are boundary points of B still $\mu p_j \in B$ for some $\mu > 1$ (there may be protruding segments). However, as B is compact and star-shaped there exist q_j on the boundary of B, arbitrarily close to the respective p_j, for which $\mu q_j \notin B$ for all $\mu > 1$. In view of (5.3) the q_j may be chosen close enough to be p_j so that we have

$$\left\| \sum_{j=1}^{k} \lambda_j q_j \right\| < 2 \left(\sum_{j=1}^{k} \lambda_j^2 \right)^{1/2} \tag{5.4}$$

for all real $\lambda_1, \ldots, \lambda_k$.

We now choose x_n for $k_\nu \leqq n < k_{\nu+1}$ so that it is a positive multiple, $x_n = \mu_j q_j$ say, of $q_{n-k_\nu+1}$ on the boundary of $B^*(\gamma_n)$ and that (2.4) holds. Then we have by (2.6)

$$\mu_j \leqq \phi\left(\frac{\gamma_n}{\rho_\nu}\right) \qquad (j = 1, \dots, k_{\nu+1} - k; \quad n = k_\nu + j - 1). \tag{5.5}$$

From (5.2), (5.4) and (5.5) we obtain

$$\left\| \sum_{n=k_{..}}^{k_{\nu+1}-1}{}' x_n \right\| < 1 \qquad (\nu = 1, 2, \dots)$$

where $'$ denotes summation on any subset. By (4.8) and (5.2) we have then

$$d\left(\sum_{n=k_\nu}^{k_{\nu+1}-1}{}' x_n, 0\right) \leqq \rho_\nu < \alpha_\nu \qquad (\nu = 1, 2, \dots).$$

This is exactly (3.7) and the proof is achieved in the same way as that of Theorem 1.

6. **Proof of Theorem** 3. We start with a second corollary of Theorem 2 which is of independent interest.

COROLLARY 2. *Let X be a complete infinite-dimensional metric space with a p-homogeneous norm* $(0 < p \leqq 1)$, *i.e. satisfying*

$$\| \lambda x \| = | \lambda |^p \| x \| \qquad \text{for all scalars } \lambda. \tag{6.1}$$

Then, given any sequence of positive numbers γ_n $(n = 1, 2, \dots)$ satisfying

$$\sum_{n=1}^{\infty} \gamma_n^{2/p} < \infty, \tag{6.2}$$

there exists in X an unconditionally convergent series (1.1) *for which* (2.9) *holds.*

Indeed, (6.1) implies $B^*(\lambda\rho) = \lambda^{1/p} B^*(\rho)$ for $\lambda > 0$. Thus X has property $C^*(\phi)$ with $\phi(\lambda) = \lambda^{1/p}$ and (6.2) is equivalent to (2.7).

On the other hand, taking $V = B^*(\rho)$ we have, by (1.3) and (6.1),

$$M_V(x) = (\| x \| / \rho)^{1/p}.$$

Therefore

$$\sum_{n=1}^{\infty} \| x_n \|^{1/p} < \infty \tag{6.3}$$

is a necessary and sufficient condition for the absolute convergence of (1.1).

It follows that in every complete infinite-dimensional space with a p-homogeneous norm there exist series which are unconditionally yet not absolutely convergent (take *e.g.* $\gamma_n = n^{-p}$ $(n = 1, 2, \dots)$).

The theorem now follows from the fact that the topology of a locally bounded space can always be given by a p-homogeneous norm with a suitable $0 < p \leqq 1$. (S. Rolewicz [7], see e.g. [6] p. 165).

7. **Proof of Theorem 4.** It may be assumed without loss of generality that the topology is given by a translation invariant metric.

Let y_1 be an arbitrary non-zero point of the space, and having determined y_i chose $y_{i+1} \neq 0$ so that

$$\Delta(y_{i+1}) < \frac{1}{2}\| y_i \|, \qquad (i = 1,2,\ldots). \tag{7.1}$$

Given any sequence of numbers ζ_i $(i=1,2,\ldots)$, it follows from (7.1) that the series

$$\sum_{i=1}^{\infty} \zeta_i y_i \tag{7.2}$$

is metrically, hence absolutely, convergent. Therefore, by completeness, it represents a point in the space.

Let Y be the totality of points representable by series (7.2). It is obviously a linear set and the representation (7.2) is unique. Indeed, we claim that if (7.2) has the value zero then all ζ_i vanish. Assume $\zeta_i = 0$ for $i < j$, $(j \geqq 1)$; if $\zeta_j \neq 0$ then

$$\frac{1}{\zeta_j} \sum_{i=j}^{\infty} \zeta_i y_i = y_j + \sum_{i=j+1}^{\infty} \frac{\zeta_i}{\zeta_j} = 0.$$

contradicting (7.1). Thus $\zeta_i = 0$ $(i = 1,2,\ldots)$.

Moreover, Y is a closed subspace. Indeed, let

$$z_n = \sum_{i=1}^{\infty} \zeta_{n,i} y_i \quad (n = 1,2,\ldots) \tag{7.3}$$

be a sequence of points of Y converging (in the original space) to z. The z_n form then a Cauchy sequence and, as in the proof uniqueness above, it follows that each sequence $\zeta_{1,i}, \zeta_{2,i}, \ldots, \zeta_{n,i} \ldots$ is a Cauchy sequence. Hence $\lim_{n=\infty} \zeta_{n,i}$ exists $(i = 1,2,\ldots)$. If we denote this limit by ζ_i then it follows immediately from (7.1) that the sequence (7.3) tends to the point represented by (7.2). Hence $z \in Y$ and Y is closed, therefore also complete.

We have in fact shown that a sequence (7.3) is convergent if and only if all the sequences $\zeta_{1,i}, \zeta_{2,i}, \ldots, \zeta_{n,i}, \ldots$ $(i = 1,2,\ldots)$ are convergent. Therefore, a series (1.1) with

$$x_n = \sum_{i=1}^{\infty} \xi_{n,i} y_i \tag{7.4}$$

is unconditionally convergent if and only if all the series

$$\sum_{n=1}^{\infty} \xi_{n\,i}, \quad (i = 1,2,\ldots) \tag{7.5}$$

are unconditionally, hence absolutely, convergent. But if the series (7.5) are absolutely convergent it follows that the Minkowski functional (1.3) with

$$V = \{x : \|x\| \leqq \|y_j\|\}$$

satisfies

$$\sum_{n=1}^{\infty} M_V(x_n) \leqq \sum_{n=1}^{\infty} \sum_{i=1}^{j} |\xi_{n,i}| \, \|y_i\| < \infty.$$

Since $\|y_j\| \to 0$ by (7.1) it follows that the unconditional convergence of (1.1) implies its absolute convergence and the theorem is established.

8. **Proof of Theorem** 5. Since X is metrizable we can define its topology by a translation invariant metric d'. We now introduce another metric d, equivalent to d', having the required properties.

Let $g(t)$ be strictly increasing and continuous for $0 \leqq t < \infty$ with $g(0) = 0$ and such that

$$\sum_{n=1}^{\infty} \gamma_n g(\gamma_n) < \infty. \tag{8.1}$$

Put $f(t) = tg(t)$. Then $f(t)$ is continuous and strictly increasing from 0 to ∞ and we have for $0 < s, t < \infty$

$$g(s + t) \geqq \max(g(s), g(t)) \geqq \frac{s}{s+t} g(s) + \frac{t}{s+t} g(t),$$

or

$$f(s + t) \geqq f(s) + f(t); \tag{8.2}$$

and (8.2) obviously remains valid when either s or t, or both, vanish. Let f^{-1} be the inverse function of f and define

$$d(x, y) = f^{-1}(d'(x, y))$$

for all $x, y \in X$.

Then $d(x,y)$ is again translation invariant and induces the same topology as $d'(x,y)$. But $\|x_n\| \leqq \gamma_n$ implies $\|x_n\|' \leqq f(\gamma_n) = \gamma_n g(\gamma_n)$. Thus, by (8.1), (2.10) implies the metric convergence of (1.1)—relative to $\| \quad \|'$—and hence its absolute convergence.

9. **Remarks.** 9.1. The requirement of completenes can be dropped throughout if unconditional convergence is replaced by Cauchy unconditional convergence (i.e. the partial sums of every rearranged series from a Cauchy sequence).

9.2. The contraction condition $C^*(\phi)$ in Theorem 2 can be weakened to the following one: There exists a null-sequence of positive numbers ρ_N ($N = 1, 2, \ldots$) and corresponding to each ρ_N subspaces Y_N of arbitrary high finite dimension such that

$$Y_N \cap B^*(\lambda\rho_N) \subset Y_N \cap \phi(\lambda)B^*(\rho_N) \text{ for } N = 1,2,\ldots \text{ and } 0 < \lambda < 1.$$

Indeed the only modification in the proof is that the ρ_ν in (5.2) have to be chosen from the given sequence and the m-dimensional Y from the corresponding subspaces.

9.3. Further results can be deduced if one couples $C^*(\phi)$, with conditions of the type $B^*(\lambda\rho) \supset \psi(\lambda)B^*(\rho)$. (A similar remark applies to 9.2 as well as to further weakenings of $C^*(\phi)$).

This is particularly striking when $B^*(\lambda\rho) = \phi(\lambda)B^*(\rho)$ as in the case of Corollary 2. Then the gain due to the second-power of λ_j in (4.7) is most evident.

9.4. If $\| \ \|$ is the ordinary norm in Hilbert space and we introduce a p-homogeneous norm by $\|x\|' = \|x\|^p$ $(0 < p \leqq 1)$ we see from the fact that $\sum \|x_n\|^2 < \infty$ is necessary for unconditional convergence of (1.1) in Hilbert space that the result of Corollary 2 is best possible.

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